Example 1. *Find a real root of the equation $3x = \cos x + 1$ by Newton-Raphson's method.*

Solution. Let $f(x) = 3x - \cos x - 1 = 0$ ...(1)

then $f(0) = -2$

and $f(1) = 3 - \cos 1 - 1$

$= 3 - 0.5403 - 1 = 1.4597.$

So a root of the equation $f(x) = 0$ lies between 0 and 1.

Let us take $x_0 = 0.6.$

From (1) $f'(x) = 3 + \sin x.$ ...(2)

Therefore, the Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \quad \text{[From (1) and (2)]}$$

$$\therefore \quad x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}. \quad \text{...(3)}$$

Putting $n = 0$, we get first approximation

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$$= \frac{0.6 \sin (0.6) + \cos (0.6) + 1}{3 + \sin (0.6)}$$

$$= \frac{0.6\,(0.5646) + 0.8253 + 1}{3 + 0.5646}$$

$$= \frac{2.16406}{3.5646} = 0.6071.$$

Putting $n = 1$ into (3), we get second approximation which is given by

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{0.6071 \sin (0.6071) + \cos (0.6071) + 1}{3 + \sin (0.6071)}$$

$$= \frac{0.6071\,(0.5705) + 0.8213 + 1}{3 + 0.5705}$$

$$= \frac{2.1677}{3.5705} = 0.6071.$$

Here $x_1 = x_2$. So the root correct to four decimal places is 0.6071.

Example 2. *Find the real root of the equation $x \log_{10} x = 1.2$ by Newton-Raphson's Method.*

Solution. Let $f(x) = x \log_{10} x - 1.2 = 0$ ...(1)

Then $f(1) = -1.2$

and $f(2) = 2 \log_{10} 2 - 1.2 = -0.5979$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.2314.$

Thus, a root of $f(x) = 0$ lies between 2 and 3.

Let us take $x_0 = 2$

From (1)

$$f'(x) = \log_{10} x + \frac{1}{x} \cdot x \log_{10} e$$

$$= \log_{10} x + 0.4343. \quad \text{...(2)}$$

Now, Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.4343} \quad \text{[From (1) and (2)]}$$

$$x_{n+1} = \frac{0.4343\, x_n + 1.2}{\log_{10} x_n + 0.4343}, \quad n = 0, 1, 2, \ldots \quad \text{...(3)}$$

Putting $n = 0$, we get the first approximation as

$$x_1 = \frac{0.4343\, x_0 + 1.2}{\log_{10} x_0 + 0.4343}$$

$$= \frac{0.4343\,(2) + 1.2}{\log_{10} 2 + 0.4343}$$

$$= \frac{2.0686}{0.7353} = 2.8133.$$

Putting $n = 1$ into (3), we get second approximation as

$$x_2 = \frac{0.4343\, x_1 + 1.2}{\log_{10} x_1 + 0.4343}$$

$$= \frac{0.4343\,(2.8133) + 1.2}{\log_{10} 2.8133 + 0.4343}$$

$$= \frac{2.4128}{0.8835} = 2.7411.$$

Putting $n = 2$ into (3), we get third approximation as

$$x_3 = \frac{0.4343\, x_2 + 1.2}{\log_{10} x_2 + 0.4343}$$

$$= \frac{0.4343\,(2.7411) + 1.2}{\log_{10} 2.7411 + 0.4343}$$

$$= \frac{2.3905}{0.8722} = 2.7408.$$

Putting $n = 3$ into (3), we get fourth approximation

$$x_4 = \frac{0.4343\, x_3 + 1.2}{\log_{10} x_3 + 0.4343}$$

$$= \frac{0.4343\,(2.7408) + 1.2}{\log_{10} 2.7408 + 0.4343}$$

$$= \frac{2.3903}{0.8721} = 2.7408.$$

Here $x_3 = x_4$, so that the root of $f(x) = 0$ correct to four decimal places is 2.7408.

Example 3. *Find the real root of the equation* $x^2 - 5x + 2 = 0$ *by Newton-Raphson's method.*

Solution. Let $f(x) \equiv x^2 - 5x + 2 = 0$ ...(1)

Then, $f(4) = 4^2 - 5\,(4) + 2 = -2$

and $f(5) = 5^2 - 5\,(5) + 2 = 2.$

Therefore, the root lies between 4 and 5.

From (1) $f'(x) = 2x - 5.$ ...(2)

Thus, Newton-Raphson's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 5x_n + 2}{2x_n - 5} \qquad \text{[From (1) and (2)]}$$

or

$$x_{n+1} = \frac{x_n^2 - 2}{2x_n - 5}, \quad n = 0, 1, 2, \ldots . \qquad \text{...(3)}$$

Let us take $x_0 = 4.$

Putting $n = 0$ into (3), we get first approximation to the root as

$$x_1 = \frac{x_0^2 - 2}{2x_0 - 5} = \frac{4^2 - 2}{2\,(4) - 5}$$

$$= \frac{14}{3} = 4.6667.$$

Putting $n = 1$ into (3), we get second approximation as

$$x_2 = \frac{x_1^2 - 2}{2x_1 - 5}$$

$$= \frac{(4.6667)^2 - 2}{2\,(4.6667) - 5} = \frac{19.7781}{4.3334}$$

$$= 4.5641.$$

Putting $n = 2$, into (3), we get third approximation as

$$x_3 = \frac{x_2^2 - 2}{2x_2 - 5}$$

$$= \frac{(4.5641)^2 - 2}{2\,(4.5641) - 5} = \frac{18.8310}{4.1282}$$

$$= 4.5616.$$

Putting $n = 3$ into (3), we get fourth approximation as

$$x_4 = \frac{x_3^2 - 2}{2x_3 - 5}$$

$$= \frac{(4.5616)^2 - 2}{2\,(4.5616) - 5} = \frac{18.8082}{4.1232}$$
$$= 4.5616.$$

Here, $x_3 = x_4$. Thus the root of the equation correct to four decimal places is 4.5616.

Example 4. *Find the real root of the equation $x^4 - x - 10 = 0$, correct to three decimal places by Newton-Raphson's method.*

Solution. Let $\quad f(x) \equiv x^4 - x - 10 = 0. \qquad ...(1)$

Then $\quad f(1) = (1)^4 - 1 - 10 = -10$

and $\quad f(2) = (2)^4 - 2 - 10 = 4.$

Therefore, the root lies between 1 and 2.

From (1) $\quad f'(x) = 4x^3 - 1. \qquad ...(2)$

Then, by Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} \qquad \text{[From (1) and (2)]}$$

$$\therefore \quad x_{n+1} = \frac{3x_n^4 + 10}{4x_n^3 - 1}, \quad n = 0, 1, 2, \ldots . \qquad ...(3)$$

Let us assume $x_0 = 1{\cdot}6$.

Now putting $n = 0$ into (3), we get first approximation as

$$x_1 = \frac{3x_0^4 + 10}{4x_0^3 - 1}$$
$$= \frac{3\,(1{\cdot}6)^4 + 10}{4\,(1{\cdot}6)^3 - 1} = \frac{29{\cdot}6608}{15{\cdot}384}$$
$$= 1{\cdot}92803.$$

Putting $n = 1$ into (3), we get second approximation as

$$x_2 = \frac{3x_1^4 + 10}{4x_1^3 - 1}$$
$$= \frac{3\,(1{\cdot}92803)^4 + 10}{4\,(1{\cdot}92803)^3 - 1} = \frac{51{\cdot}45495}{27{\cdot}66826}$$
$$= 1{\cdot}85971.$$

Putting $n = 2$ into (3), we get third approximation as

$$x_3 = \frac{3x_2^4 + 10}{4x_2^3 - 1}$$
$$= \frac{3\,(1{\cdot}85971)^4 + 10}{4\,(1{\cdot}85971)^3 - 1} = \frac{45{\cdot}88411}{24{\cdot}72739}$$
$$= 1{\cdot}85559.$$

Putting $n = 3$ into (3), we get fourth approximation as

$$x_4 = \frac{3x_3^4 + 10}{4x_3^3 - 1}$$

$$= \frac{3\,(1{\cdot}85559)^4 + 10}{4\,(1{\cdot}85559)^3 - 1} = \frac{45{\cdot}56717}{24{\cdot}55677}$$

$$= 1{\cdot}85558.$$

Hence, the required root correct to three of decimal is 1·856.

Example 5. *Using Newton-Raphson's method. Find the square root of 12 correct to three places of decimal.*

Solution. Let $\qquad f(x) \equiv x^2 - 12 = 0 \qquad$ ...(1)

then $\qquad f(1) = -11$

and $\qquad f(2) = -8$

and $\qquad f(3) = -3$

and $\qquad f(4) = 4.$

∴ The root lies between 3 and 4.

Further, $\qquad f(3{\cdot}1) = (3{\cdot}1)^2 - 12 = -2{\cdot}39$

$$f(3{\cdot}2) = (3{\cdot}2)^2 - 12 = -1{\cdot}76$$

$$f(3{\cdot}3) = -1{\cdot}11$$

$$f(3{\cdot}4) = -0{\cdot}44$$

$$f(3{\cdot}5) = 0{\cdot}25.$$

Thus, the required root lies between 3·4 and 3·5.

Now, from (1), $\qquad f'(x) = 2x.$

Then, Newton-Raphson's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{...(2)}$$

$$= x_n - \frac{x_n^2 - 12}{2x_n} \qquad \text{[From (1) and (2)]}$$

or

$$x_{n+1} = \frac{x_n^2 + 12}{2x_n},\ n = 0, 1, 2, \ldots \qquad \text{...(3)}$$

Let us assume that $\qquad x_0 = 3{\cdot}4.$

Putting $n = 0$ into (3), we get first approximation as

$$x_1 = \frac{x_0^2 + 12}{2x_0}$$

$$= \frac{(3{\cdot}4)^2 + 12}{2\,(3{\cdot}4)} = \frac{23{\cdot}56}{6{\cdot}8} = 3{\cdot}4647.$$

Putting $n = 1$ into (3), we get second approximation as

$$x_2 = \frac{x_1^2 + 12}{2x_1} = \frac{(3{\cdot}4647)^2 + 12}{2\,(3{\cdot}4647)}$$

$$= \frac{24{\cdot}0041}{6{\cdot}9294} = 3{\cdot}4641.$$

Putting $n = 2$ into (3), we get third approximation as

$$x_3 = \frac{x_2^2 + 12}{2x_2} = \frac{(3.4641)^2 + 12}{2\,(3.4641)}$$

$$= \frac{23.9999}{6.9282} = 3.4641.$$

Hence the required root correct to three places of decimal is 3.464.

Example 6. *Find the real root of the equation $\log x - \cos x = 0$ correct to three places of decimal by Newton–Raphson's method.*

Solution. Let $\quad f(x) \equiv \log x - \cos x = 0$ ...(1)

then $\quad f(1) = -0.5403$

and $\quad f(2) = \log 2 - \cos 2$

$= 0.6931 + 0.4161 = 1.1092.$

$\therefore$ The root lies between 1 and 2.

Further, $\quad f(1.1) = \log 1.1 - \cos 1.1$

$= 0.0953 - 0.4535 = -0.3582$

and $\quad f(1.2) = \log 1.2 - \cos 1.2$

$= 0.1823 - 0.3623 = -0.18$

and $\quad f(1.3) = \log 1.3 - \cos 1.3$

$= 0.2623 - 0.2674 = -0.0051$

and $\quad f(1.4) = \log 1.4 - \cos 1.4$

$= 0.3364 - 0.1699 = 0.1665.$

Thus, the required root will lie between 1.3 and 1.4.

Now, from (1) $\quad f'(x) = \frac{1}{x} + \sin x.$ ...(2)

Then, by Newton–Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\log x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n} \qquad \text{[From (1) and (2)]}$$

$$x_{n+1} = \frac{x_n + x_n^2 \sin x_n - x_n \log x_n + x_n \cos x_n}{1 + x_n \sin x_n} \qquad ...(3)$$

Let us assume that $x_0 = 1.3$.

Now putting $n = 0$ into (3), we get first approximation as

$$x_1 = \frac{x_0 + x_0^2 \sin x_0 - x_0 \log x_0 + x_0 \cos x_0}{1 + x_0 \sin x_0}$$

$$= \frac{1.3 + (1.3)^2 \sin(1.3) - 1.3 \log(1.3) + 1.3 \cos(1.3)}{1 + 1.3 \sin(1.3)}$$

$$= \frac{1.3\,(1 + 1.2526 - 0.2623 + 0.2674)}{1 + 1.2526}$$

$$= \frac{2.93501}{2.2526} = 1.3029.$$

Putting $n = 1$ into (3), we get second approximation as

$$x_2 = \frac{x_1 (1 + x_1 \sin x_1 - \log x_1 + \cos x_1)}{1 + x_1 \sin x_1}$$

$$= \frac{1.3029 (1 + 1.3029 \sin 1.3029 - \log 1.3029 + \cos 1.3029)}{1 + 1.3029 \sin 1.3029}$$

$$= \frac{1.3029 (1 + 1.2564 - 0.2645 + 0.2647)}{1 + 1.2564}$$

$$= \frac{2.9401}{2.2564} = 1.3030.$$

Hence, the required root correct to three decimal places is 1.303.

Example 7. *Find a positive value of* $(17)^{1/3}$ *correct to four decimal places by Newton-Raphson's method.*

Solution. General formula for p^{th} root of 'a' is given by

$$x_{n+1} = \frac{(p-1) x_n^p + a}{p x_n^{p-1}}. \qquad \text{...(1)}$$

According to the question,

$$a = 17, p = 3.$$

So (1) becomes

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{17}{x_n^2}\right), \ n = 0, 1, 2, \dots . \qquad \text{...(2)}$$

Since $(8)^{1/3} = 2$ and $(27)^{1/3} = 3$ and $8 < 17 < 27$ so that the initial approximation is taken as

$$x_0 = \frac{2+3}{2} = 2{\cdot}5.$$

For $n = 0$, we get

$$x_1 = \frac{1}{3}\left(2x_0 + \frac{17}{x_0^2}\right) = \frac{1}{3}\left(5 + \frac{17}{6{\cdot}25}\right)$$

$$= \frac{1}{3}(7{\cdot}72) = 2{\cdot}5733.$$

For $n = 1$, we get

$$x_2 = \frac{1}{2}\left(2x_1 + \frac{17}{x_1^2}\right)$$

$$= \frac{1}{3}\left[2\,(2{\cdot}5733) + \frac{17}{6{\cdot}6219}\right]$$

$$= \frac{1}{3}(7{\cdot}7138) = 2{\cdot}5713.$$

For $n = 2$, we get

$$x_3 = \frac{1}{3}\left(2x_2 + \frac{17}{x_2^2}\right)$$

$$= \frac{1}{3}\left[2\,(2{\cdot}5713) + \frac{17}{6{\cdot}61158}\right] = \frac{1}{3}(7{\cdot}71385)$$

$$= 2{\cdot}57128.$$

For $n = 3$, we get

$$x_4 = \frac{1}{3}\left(2x_3 + \frac{17}{x_3^2}\right)$$

$$= \frac{1}{3}\left[2\,(2{\cdot}57128) + \frac{17}{6{\cdot}61148}\right] = \frac{1}{3}\,(7{\cdot}71385)$$

$$= 2{\cdot}57128.$$

Thus, x_3 and x_4 approximations are equal, so that the required root correct to four decimal places is 2·5713.

Example 8. *Determine the values of p and q so that the rate of convergence of the iterative method*

$$x_{n+1} = px_n + q\,\frac{N}{x_n^2}$$

for computing $(N)^{1/3}$ *becomes as high as possible.*

Solution. Let $f(x) \equiv x^3 - N = 0$ and let ξ be the exact root of $f(x) = 0$, then

$$f(\xi) = 0 \quad \text{or} \quad \xi^3 = N.$$

Let ε_n be the error in the n^{th} approximation of ξ, then putting

$$x_n = \xi + \varepsilon_n \text{ and } x_{n+1} = \xi + \varepsilon_{n+1}$$

in the given iterative method, we get

$$\xi + \varepsilon_{n+1} = p(\xi + \varepsilon_n) + q\,\frac{\xi^3}{(\xi + \varepsilon_n)^2}$$

$$= p(\xi + \varepsilon_n) + q\,\frac{\xi^3}{\xi^2\left(1 + \frac{\varepsilon_n}{\xi}\right)^2}$$

$$= p(\xi + \varepsilon_n) + q\xi\left(1 + \frac{\varepsilon_n}{\xi}\right)^{-2}$$

$$= p(\xi + \varepsilon_n) + q\xi\left[1 - 2\,\frac{\varepsilon_n}{\xi} + 3\left(\frac{\varepsilon_n}{\xi}\right)^2 - \ldots\right]$$

$$= p(\xi + \varepsilon_n) + q\xi - 2q\varepsilon_n + 2q\,\frac{\varepsilon_n^2}{\xi} - \ldots$$

$$= p\xi + q\xi + (p - 2q)\,\varepsilon_n + 0(\varepsilon_n^2) + \ldots$$

or

$$\varepsilon_{n+1} = (p + q - 1)\,\xi + (p - 2q)\,\varepsilon_n + 0\,(\varepsilon_n^2) + \ldots.$$

For the given method to become of order as high as possible *i.e.*, of order 2, we must have

$$p + q - 1 = 0 \text{ and } p - 2q = 0.$$

on solving, we get

$$p = \frac{2}{3} \text{ and } q = \frac{1}{3}.$$

Example 9. *Show that the square root of* $N = AB$ *is given by*

$$\sqrt{N} = \frac{S}{4} + \frac{N}{S}$$

where $S = A + B.$

Solution. Let $f(x) = x^2 - N = 0$ ...(1)

then $f'(x) = 2x.$

By Newton-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

or
$$x_{n+1} = \frac{x_n}{2} + \frac{N}{2x_n}.$$

Let $x_n = \frac{A+B}{2}$, then

$$x_{n+1} = \frac{A+B}{4} + \frac{N}{A+B}$$

$$= \frac{S}{4} + \frac{N}{S} \qquad [\because \ S = A + B]$$

$\therefore$
$$\sqrt{N} = \frac{S}{4} + \frac{N}{S}.$$

Example 10. *Show that the following two sequences, both have convergence of the second order with the same limit* $\sqrt{a}$:

$$x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right) \text{ and } x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right).$$

Solution. Let ε_n be the error at the n^{th} approximation to the root $\sqrt{a}$. Then

$$x_n = \sqrt{a} + \varepsilon_n \text{ and } x_{n+1} = \sqrt{a} + \varepsilon_{n+1}.$$

Since $x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right)$, we have

$$\sqrt{a} + \varepsilon_{n+1} = \frac{1}{2} (\sqrt{a} + \varepsilon_n) \left(1 + \frac{a}{(\sqrt{a} + \varepsilon_n)^2}\right)$$

$$= \frac{1}{2} (\sqrt{a} + \varepsilon_n) \left[1 + \left(1 + \frac{\varepsilon_n}{\sqrt{a}}\right)^{-2}\right]$$

$$= \frac{1}{2} (\sqrt{a} + \varepsilon_n) \left[1 + 1 - 2\frac{\varepsilon_n}{\sqrt{a}} + 3\left(\frac{\varepsilon_n}{\sqrt{a}}\right)^2 - \ldots\right]$$

$$= \frac{1}{2} \left[2\sqrt{a} + 2\varepsilon_n - 2\varepsilon_n - \frac{2}{\sqrt{a}} \varepsilon_n^2 + \frac{3}{\sqrt{a}} \varepsilon_n^2 + \frac{3}{a} \varepsilon_n^3 + \ldots\right]$$

$$= \sqrt{a} + \frac{1}{2\sqrt{a}} \varepsilon_n^2 + 0\,(\varepsilon_n^3) + \ldots.$$

$\therefore$
$$\varepsilon_{n+1} = \frac{1}{2\sqrt{a}} \varepsilon_n^2 + 0\,(\varepsilon_n^3) + \ldots$$

$\Rightarrow$
$$\varepsilon_{n+1} \propto \varepsilon_n^2$$

which shows that the first iterative formula has quadratic rate of convergence with limit $\sqrt{a}$.

Similarly for second iterative formula, we have

$$\sqrt{a} + \varepsilon_{n+1} = \frac{1}{2}(\sqrt{a} + \varepsilon_n)\left[3 - \frac{(\sqrt{a} + \varepsilon_n)^2}{a}\right]$$

$$= \frac{1}{2}(\sqrt{a} + \varepsilon_n)\left[3 - \frac{a + 2\sqrt{a}\,\varepsilon_n + \varepsilon_n^2}{a}\right]$$

$$= \frac{1}{2}(\sqrt{a} + \varepsilon_n)\left(2 - \frac{2}{\sqrt{a}}\varepsilon_n - \frac{1}{a}\varepsilon_n^2\right)$$

$$= \frac{1}{2}\left(2\sqrt{a} + 2\varepsilon_n - 2\varepsilon_n - \frac{2}{\sqrt{a}}\varepsilon_n^2 - \frac{1}{\sqrt{a}}\varepsilon_n^2 - \frac{1}{a}\varepsilon_n^3\right)$$

$$= \sqrt{a} - \frac{3}{2\sqrt{a}}\varepsilon_n^2 - \frac{1}{2a}\varepsilon_n^3.$$

$$\Rightarrow \qquad \varepsilon_{n+1} = \frac{3}{2\sqrt{a}}\varepsilon_n^2 - \frac{1}{2a}\varepsilon_n^3 = -\frac{3}{2\sqrt{a}}\varepsilon_n^2 + 0(\varepsilon_n^3)$$

$$\Rightarrow \qquad \varepsilon_{n+1} \propto \varepsilon_n^2.$$

This shows that the second iterative formula has also quadratic rate of convergence with limit $\sqrt{a}$.

Example 11. *Use synthetic division to solve $f(x) = x^3 - x^2 - 1{\cdot}0001x + 0{\cdot}9999 = 0$ in the neighbourhood of $x = 1$.*

Solution. To find $f(1)$ and $f'(1)$, we proceed as follows :

1	1	– 1	– 1·0001	0·9999
		1	0	– 1·0001
	1	0	– 1·0001	– 0·0002 = $f(1)$
		1	1	
	1	1	– 0·0001 = $f'(1)$	
		1		
	1	$2 = \frac{1}{2!}f''(1)$		
	1			

From above synthetic division, we observe that $f(1) = -0{\cdot}0002$ and $f'(1) = 0{\cdot}0001$, both are small, therefore there exist two nearly equal roots.

So taking $x_0 = 1$, we shall use the formula

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

For this we shall require $f''(1)$. From the above synthetic division, we have

$$2 = \frac{1}{2!}f''(1) \Rightarrow f''(1) = 4.$$

$$\therefore \qquad x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 1 - \frac{f'(1)}{f''(1)} = 1 - \frac{-0{\cdot}0001}{4} = 1{\cdot}000025.$$

Now we again calculate $f'(x_1)$ and $f''(x_1)$ by synthetic division as follows :

1.000025	1	− 1	− 1.0001	0.9999
		1.000025	0.000025	− 1.000095
	1	0.000025	− 1.000075	$-0.000195 = f(x_1)$
		1.000025	1.000075	
	1	1.000050	$0 = f'(x_1)$	
		1.000025		
	1	$2.000075 = \frac{1}{2!} f''(x_1)$		
	1			

From second synthetic division table, we have

$$f(x_1) = -0.000195 \text{ i.e., } f(1.000025) = -0.000195$$

and

$$f''(x_1) = 4.000150 \text{ i.e., } f''(1.000025) = 4.000150.$$

Thus, for nearly equal roots, the starting values are

$$x = c \pm \sqrt{\frac{-2f(c)}{f''(c)}}, \quad c = 1.000025$$

$$= 1.000025 \pm \sqrt{\frac{-2(-0.000195)}{4.000150}}$$

$$= 1.000025 \pm 0.00987402$$

$$= 1.009899, 0.990151.$$

Now taking $x = 1.009899$ and $x = 0.990151$ as starting values of two roots, we can find the two close roots by original Newton's method.

Example 12. *Find the double root of the equation* $x^3 - x^2 - x + 1 = 0$.

Solution. Let $\quad f(x) = x^3 - x^2 - x + 1 = 0 \quad$ ...(1)

then $\quad f'(x) = 3x^2 - 2x - 1$

and $\quad f''(x) = 6x - 2$

since $\quad f(0.9) = -1.439 < 0, f(1) = 0$ and $f(1.1) = 0.021$

so taking $x_0 = 0.9$, we have

$$x_1 = x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.9 - \frac{2(0.19)}{-0.37} = 1.003$$

and

$$x_1 = x_0 - (2-1)\frac{f'(x_0)}{f''(x_0)} = 0.9 - \frac{(-0.37)}{3.4} = 1.009.$$

The closeness of these values indicates that there is a double root near $x = 1$. Choosing $x_1 = 1.01$ for the next approximation, we get

$$x_2 = x_1 - 2\frac{f(x_1)}{f'(x_1)}$$

$$= 1.01 - \frac{2(0.0002)}{0.0403} = 1.01 - 0.0099 = 1.0001$$

and

$$x_2 = x_1 - (2-1)\frac{f'(x_1)}{f''(x_1)}$$

$$= 1{\cdot}01 - \frac{0{\cdot}0403}{4{\cdot}06} = 1{\cdot}01 - 0{\cdot}0099 = 1{\cdot}0001$$

we conclude that there is a double root at $x = 1{\cdot}0001$ which is very close to $x = 1$.

Example 13. *Show that the equation $f(x) = 1 - xe^{1-x} = 0$, has a double root at $x = 1$. The root is obtained by using the modified Newton-Raphson method with $m = 2$ starting with $x_0 = 0$.*

Solution. Since $f(x) = 1 - xe^{1-x} = 0$

then $f'(x) = xe^{1-x} - e^{1-x}$.

By modified Newton-Raphson method for $m = 2$, we have

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

$$= x_n - 2\frac{(1 - x_n e^{1-x_n})}{x_n e^{1-x_n} - e^{1-x_n}}$$

$$= \frac{x_n^2 e^{1-x_n} - x_n e^{1-x_n} - 2 + 2x_n e^{1-x_n}}{e^{1-x_n}(x_n - 1)}$$

$$x_{n+1} = \frac{x_n^{\ 2} e^{1-x_n} + x_n e^{1-x_n} - 2}{(x_n - 1) e^{1-x_n}}$$

or
$$x_{n+1} = \frac{(x_n^2 + x_n) e^{1-x_n} - 2}{(x_n - 1) e^{1-x_n}}, \quad n = 0, 1, 2, \ldots \qquad ...(1)$$

For $n = 0$, we get

$$x_1 = \frac{(x_0^{\ 2} + x_0) e^{1-x_0} - 2}{(x_0 - 1) e^{1-x_0}}$$

$$= \frac{-2}{-e} \qquad [\because \ x_0 = 0]$$

$$= \frac{2}{e} = 0{\cdot}735758882.$$

For $n = 1$, we get

$$x_2 = \frac{(x_1^{\ 2} + x_1) e^{1-x_1} - 2}{(x_1 - 1) e^{1-x_1}}$$

$$= \frac{(1{\cdot}277100014)(1{\cdot}302442201) - 2}{(-0{\cdot}264241118)(1{\cdot}302442201)}$$

$$= 0{\cdot}978185253.$$

For $n = 2$, we get

$$x_3 = \frac{(x_2^{\ 2} + x_2) e^{1-x_2} - 2}{(x_2 - 1) e^{1-x_2}}$$

$$= \frac{(1{\cdot}935031642)(1{\cdot}022054428) - 2}{(-0{\cdot}021814747)(1{\cdot}022054428)}$$

$$= 0{\cdot}999842233.$$

For $n = 3$, we get

$$x_4 = \frac{(x_3^2 + x_3)\, e^{1 - x_3} - 2}{(x_3 - 1)\, e^{1 - x_3}}$$

$$= \frac{(1.999526724)\,(1.000157779) - 2}{(-0.000157767)\,(1.000157779)}$$

$$= 1.000000061.$$

For $n = 4$, we get

$$x_5 = \frac{(x_4^2 + x_4)\, e^{1 - x_4} - 2}{(x_4 - 1)\, e^{1 - x_4}}$$

$$= \frac{(2.000000183)\,(1.000000183) - 2}{(0.000000061)\,(1.000000183)}$$

$$= 1.000000061.$$

Hence the root correct to nine decimal places is 1.000000061.

EXERCISE 4

1. Using Newton-Raphson's method, obtain a real root of the following equations :
   (i) $x^5 + 5x + 1 = 0$ (ii) $x = \sqrt{29}$ (iii) $x^4 + x^2 - 80 = 0$
   (iv) $\tan x = 4x$ (v) $x^3 - 5x + 3 = 0$ (vi) $x \sin x + \cos x = 0$
   (vii) $x^4 - x - 13 = 0$ (viii) $x^3 - 10 = 0$ (ix) $\log x - x + 3 = 0$
   (x) $e^x = 3x$.
2. Find to four places of decimal, the smallest root of the equation $e^{-x} = \sin x$.
3. Solve $x^4 - 5x^3 + 20x^2 - 40x + 60 = 0$ by Newton-Raphson's method given that all the roots of the given equation are complex.
4. Use Newton-Raphson's method to obtain a root correct three decimal places of the following equations :
   (i) $\sin x = 1 - x$ (ii) $x^3 + 3x^2 - 3 = 0$ (ii) $x + \log x = 2$
   (iv) $\tan x = x$ (v) $4\,(x - \sin x) = 1$ (vi) $x = 2 \sin x$
   (vii) $x^3 - 5x + 3 = 0$ (viii) $x^2 + 4 \sin x = 0$ (ix) $\log x = \cos x$
   (x) $x^4 + x^2 - 80 = 0$.
5. Compute one positive root of $2x - \log_{10} x = 7$ by Newton-Raphson's method correct to four decimal places.
6. Explain the method of Newton-Raphson for computing roots. Apply it for finding x from $x^2 - 25 = 0$.
7. Find the formula for the fourth root of a positive number N using Newton-Raphson's method and hence find $(32)^{1/4}$.
8. Prove the recurrence formula

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$$

   for finding the cube root of N. Hence find the cube root of 63.
9. Use Newton's formula to prove that square root of N can be obtained by the recurrence formula,

$$x_{n+1} = x_n\left(1 - \frac{x_n^2 - N}{2N}\right).$$

   Hence find the square root of 26, 29 and 35.
10. Obtain the cube root of 120 using Newton-Raphson's method starting with $x_0 = 4.5$.

11. Determine p, q and r so that the order of the iterative method

$$x_{n+1} = px_n + \frac{qa}{x_n^2} + \frac{ra^2}{x_n^5}$$

for $a^{1/3}$ becomes as high as possible.

12. How should the constant α be chosen to ensure the fastest possible convergence with the iteration formula

$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}.$$

13. By using Newton-Raphson's method, find the root of $x^4 - x - 10 = 0$ which is near to $x = 2$ correct to three decimal places.

14. Using Newton-Raphson's method find a root of $x^4 + x^3 + 5x^2 + 4x + 4 = 0$ standing $x_0 = i$.

15. Obtain Newton-Raphson's extended formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \cdot \frac{[f(x_0)]^2 \cdot f''(x_0)}{[f'(x_0)]^3}$$

for the root of the equation $f(x) = 0$ it is also known as Chebyshev formula of third order.

16. If x_n is suitable close approximation to $\sqrt{a}$ show that error in the formula

$$x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right)$$

is about $\frac{1}{3}$rd that in formula

$$x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right)$$

and deduce that the formula

$$x_{n+1} = \frac{x_n}{8}\left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$$

gives a sequence with third order convergence.

17. Show that the modified Newton-Raphson's method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

gives a quadratic convergence when $f(x) = 0$ has a pair of double root in the neighbourhood of $x = x_n$.

18. The equation $f(x) = x^3 - 7x^2 + 16x - 12 = 0$ has a double root at $x = 2$. Starting with the initial approximation $x_0 = 1$, find the root correct to three decimal places using modified Newton-Raphson's method with $m = 2$.

ANSWERS

1. (i) -0.1999 (ii) 5.3852 (iii) 2.908 (iv) 0.0000 (v) 0.6566
(vi) 2.798 (vii) 1.967 (viii) 2.1547 (ix) 0.052 (x) 0.61904

2. 0.5885 **3.** $1.915 \pm 1.908i$, $0.585 \pm 2.805i$

4. (i) 0.511 (ii) -2.533 (iii) 1.756 (iv) 4.4934 (v) 1.171
(vi) 1.896 (vii) 0.657 (viii) -1.934 (ix) 1.303 (x) 2.908

5. 3.7892 **6.** 5 **7.** $x_{n+1} = \frac{1}{4}\left(3x_n + \frac{N}{x_n^3}\right)$; 2.3784 **8.** 3.979

9. (i) 5.099 (ii) 5.384 (iii) 5.916 **10.** 4.9324

11. $p = \frac{5}{9}, q = \frac{5}{9}, r = -\frac{1}{9}$, Third order **12.** $\alpha = 0.6353$ **13.** 1.856

14. $-0.499 + 0.866i$ **18.** 2.000.